MINIMAL RESOLUTION OF GENERAL STABLE VECTOR BUNDLES ON $\mathbb{P}^2$

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Abstract. We study the general elements of the moduli spaces $\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2)$ of stable holomorphic vector bundle on $\mathbb{P}^2$ and their minimal free resolution. Incidentally, a quite easy proof of the irreducibility of $\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2)$ is shown.

1. Introduction

In this paper we investigate stable vector bundles on the complex projective plane $\mathbb{P}^2$ by means of their minimal free resolution. The fundamental background for this study is the work of Bohnhorst and Spindler [BS92] and their idea to use admissible pairs to characterize the stability of rank-n vector bundles on $\mathbb{P}^n$ and to give a stratification of the relative moduli space in constructible subsets.

Two main difficulties arise: (i) to state a weak version of the Bohnhorst-Spindler theorems for rank $r \geq 2$ vector bundles on $\mathbb{P}^2$, (ii) to estimate the codimension of the constructible subsets of the moduli space. In section 2 we make some general remarks on rank r vector bundles on $\mathbb{P}^n$ with $r \geq n$ to address (i), while (ii) is the object of the last section.

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2. Admissible pairs and resolutions

Let

$$0 \to \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^n}(-a_i) \xrightarrow{\Phi} \bigoplus_{j=1}^{r+k} \mathcal{O}_{\mathbb{P}^n}(-b_j) \to \mathcal{E} \to 0 \tag{1}$$

be a free resolution of length 1 of a rank r vector bundle $\mathcal{E}$ on $\mathbb{P}^n$. We assume that the two sequences a_i and b_i are indexed in nondecreasing order

$$\begin{aligned} &a_1 \leq a_2 \leq \cdots \leq a_k, \\ &b_1 \leq b_2 \leq \cdots \leq b_k \leq \cdots \leq b_{r+k}. \end{aligned} \tag{2}$$

We call $(a, b) = ((a_1, \ldots, a_k), (b_1, \ldots, b_{r+k}))$ the *pair* associated to the resolution (1). If the resolution (1) is minimal, then we call (a, b) the pair associate to the bundle $\mathcal{E}$. Notice that the associated pair and the Betti numbers of a resolution encode exactly the same information; in particular $\max(a_k - 1, b_{r+k})$ is the regularity.

2.1. Definition. The pair (a, b) is said to be weakly admissible if

$$a_i > b_{n+i} \qquad \text{for all } i = 1, \ldots, k \tag{3}$$

and admissible (or strongly admissible) if

$$a_i > b_{r+i} \qquad \text{for all } i = 1, \ldots, k \tag{4}$$

For brevity we say that the resolution (1) is weakly or strongly admissible if the associated pair (a, b) is.

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2.2. *Example.* The map Φ can be expressed by a $(r+k)\times k$ matrix of forms $(\phi_{i,j})$ of degree $\deg(\phi_{i,j}) = (b_i - a_j)$. If (a,b) is a strongly admissible pair and $\omega_0 \dots \omega_r$ are linear forms in general position on $\mathbb{P}^n$, then the $(r+k)\times k$ matrix

$$(\phi_{ij}) := \begin{bmatrix} \omega_0^{a_1-b_1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ \vdots & & \omega_0^{a_k-b_k} \\ \omega_r^{a_1-b_{r+1}} & & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \dots & \omega_r^{a_k-b_{r+k}} \end{bmatrix} \tag{5}$$

defines a minimal free resolution and $\mathcal{E} := \operatorname{coker}\Phi$ is a vector bundle with associated pair (a,b). If the pair (a,b) is only weakly admissible, the same reasoning works for the ω's defined by

$$(\omega_0, \dots, \omega_n, \omega_{n+1}, \dots, \omega_r) = (x_0, \dots, x_n, 0, \dots, 0)$$

Admissibility was originally introduced by Bohnhorst and Spindler in [BS92] to characterize stability for vector bundles on $\mathbb{P}^n$ of homological dimension 1 with rank r equal to n. Note that for $r = n$ weakly and strongly admissibility coincide. In this section we are going to restate their results with some generalizations to the case of rank $r \geq n$.

2.3. Proposition. *If $r \geq n$, the following two condition on the resolution (1) are equivalent:*

1. *is minimal;*
2. *is weakly admissible and every constant entry of the matrix $(\phi_{i,j})$ is zero.*

Proof. Obviously, (2.) follows from (1.). For $r = n$ the statement was proved by Bohnhorst and Spindler ([BS92] proposition 2.3). Now suppose that $r > n$ and (1) is minimal. Since $\mathcal{E}(b_{r+k})$ is globally generated, Bertini's theorem ensures that a generic map $f\colon \mathcal{O}(-b_{r+k}) \to \mathcal{E}$ is injective. Then, in the following commutative diagram columns and rows are exact and $\mathcal{E}''$ is locally free:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & \mathcal{O}(-b_{r+k}) \arrow[r, "Id"] \arrow[d] & \mathcal{O}(-b_{r+k}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \oplus_{i=1}^{k}\mathcal{O}(-a_i) \arrow[r] \arrow[d, "Id"] & \oplus_{i=1}^{r+k}\mathcal{O}(-b_i) \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \oplus_{i=1}^{k}\mathcal{O}(-a_i) \arrow[r] \arrow[d] & \oplus_{i=1}^{r+k-1}\mathcal{O}(-b_i) \arrow[r] \arrow[d] & \mathcal{E}'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd} \tag{6}$$

The minimality of the middle row yields the minimality of the last row. Using induction on r, we may assume that the last row is weakly admissible. Then, the middle row is also weakly admissible. ∎

As a consequence, to every vector bundle it corresponds a weakly admissible pair. Vice versa, for every weakly admissible pair, the examples 2.2 provide a way to construct a vector bundle with such associated pair.

2.4. Theorem (Bohnhorst-Spindler [BS92])**.** *Suppose $r = n$ and that the resolution (1) is admissible. Let $c_1 = \sum a_i - \sum b_j$ be the first Chern class and $\mu = c_1/n$ the slope of $\mathcal{E}$. Then $\mathcal{E}$ is stable (respectively semistable) if and only if*

$$b_1 > -\mu \qquad \text{(resp. } b_1 \geq -\mu\text{).} \tag{7}$$

For the case of higher rank, we have no chances to extend the above arithmetical characterization since stable and unstable vector bundles may have the same associated pair. However, one implication still hold:

2.5. Theorem. *If the resolution (1) is weakly admissible (in particular if it is minimal) and $\mathcal{E}$ is a stable (resp. semistable) vector bundle, then the associated pair (a, b) is strongly admissible and $b_1 > -\mu$ (resp. $b_1 \geq -\mu$).*

Proof. We first prove that if $\mathcal{E}$ is semistable, then $b_{r+k} < a_k$. In fact, if $b_{r+k} \geq a_k$ then $\mathcal{E}$ split as $\mathcal{E} = \mathcal{O}(-b_{k+r}) \oplus \mathcal{E}''$ and by weakly admissibility

$$\begin{aligned} \sum_{i=1}^{k} a_i - \sum_{j=1}^{r+k-1} b_j &= -\sum_{i=1}^{n} b_i + \sum_{i=1}^{k}(a_i - b_{n+i}) - \sum_{i=n+k+1}^{r+k-1} b_i \\ &\geq -nb_{r+k} + k - (r-n-1)b_{r+k} \\ &> (1-r)b_{r+k} \end{aligned} \tag{8}$$

then we have $\mu(\mathcal{E}'') > -b_{r+k} = \mu(\mathcal{O}(-b_{r+k}))$ which contradicts the semistability of $\mathcal{E}$.

Now suppose that $\mathcal{E}$ is semistable and $a_s \leq b_{r+s}$ for some s with $1 \leq s < k$ and let $(s_0 - 1)$ be the largest of such s. Since

$$a_1 \geq \cdots \geq a_{s_0} \geq b_{r+s_0} \geq \cdots \geq b_{r+k},$$

the minor Φ'' of Φ, obtained by cutting off the last $(k - s_0)$ rows and $(k - s_0)$ columns, remains of maximal rank so $\mathcal{E}'' := \operatorname{coker} \Phi''$ is a vector bundle. A surjective morphism $\mathcal{E} \to \mathcal{E}'' \to 0$ is defined by the diagram

$$\begin{tikzcd} 0 \arrow[r] & \oplus_{i=1}^{k}\mathcal{O}(-a_i) \arrow[r, "\Phi"] \arrow[d] & \oplus_{j=1}^{r+k}\mathcal{O}(-b_j) \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \oplus_{i=1}^{s_0}\mathcal{O}(-a_i) \arrow[r, "\Phi''"] & \oplus_{j=1}^{r+s_0}\mathcal{O}(-b_j) \arrow[r] & \mathcal{E}'' \arrow[r] & 0 \end{tikzcd} \tag{9}$$

where the first two vertical map are the natural projections. Observe that $\mathcal{E}''$ have the same rank r as $\mathcal{E}$ and

$$\begin{aligned} \mu(\mathcal{E}'') = \frac{1}{r}\left(\sum_{i=1}^{s_0} a_i - \sum_{j=1}^{r+s_0} b_j\right) &= \mu(\mathcal{E}) - \frac{1}{r}\sum_{i=s_0+1}^{k}(a_i - b_{r+i}) \\ < \mu(\mathcal{E}). \end{aligned} \tag{10}$$

Then, $\mathcal{E}''$ must be semistable otherwise any torsionless quotient sheaf destabilizing it would also destabilize $\mathcal{E}$. By induction on k, we may assume that the second row of (9) is strongly admissible. In particular, we have $a_{s_0} > b_{r+s_0}$ that gives a contradiction.

Finally, if $\mathcal{E}$ is stable (resp. semistable), then

$$\mathrm{H}^0(\mathcal{E}(m)) = 0 \qquad \forall m \leq -\mu(\mathcal{E}) \quad (\text{resp. } \forall m < -\mu(\mathcal{E})) \tag{11}$$

but, from the exact sequence

$$0 \to \bigoplus_{i=1}^{k} \mathcal{O}(-a_i + b_1) \to \bigoplus_{j=1}^{r+k} \mathcal{O}(-b_j + b_1) \to \mathcal{E}(b_1) \to 0, \tag{12}$$

we have $\mathrm{H}^0(\mathcal{E}(b_1)) \neq 0$ then $b_1 > -\mu(\mathcal{E})$ (resp. $b_1 \geq -\mu(\mathcal{E})$). $\square$

3. On vector bundles on $\mathbb{P}^2$

By Horrocks theorem, every rank r vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ has homological dimension at most 1, that is, if $\mathcal{E}$ does not split in the direct sum of line bundles, then it is presented by a minimal free resolution of the form

$$0 \to \bigoplus_{i=1}^{k} \mathcal{O}_{\mathbb{P}^2}(-a_i) \xrightarrow{\Phi} \bigoplus_{j=1}^{r+k} \mathcal{O}_{\mathbb{P}^2}(-b_j) \to \mathcal{E} \to 0. \tag{13}$$

The Chern classes c_1, c_2 of $\mathcal{E}$ are determined by a_i and b_j with the formulas

$$\begin{aligned} c_1 &= \sum_{i=1}^{k} a_i - \sum_{i=1}^{k+r} b_i, \\ 2c_2 - c_1^2 &= \sum_{i=1}^{k} a_i^2 - \sum_{i=1}^{k+r} b_i^2. \end{aligned} \tag{14}$$

We denote by $\mathfrak{I}$ the set of all (strongly) admissible pairs (a, b) associated to rank r-vector bundles on $\mathbb{P}^2$ with Chern classes c_1, c_2 satisfying the condition $b_1 > -\mu = (\sum a_i - \sum b_j)/r$. Theorem 2.5 shows that the set $\mathfrak{I}$ contains the set of all possible associated pair to a stable vector bundle in $\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2)$ and coincides exactly with it for $r = n$. Then

$$\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2) = \coprod_{(a,b)\in\mathfrak{I}} \mathfrak{M}(a, b) \tag{15}$$

where $\mathfrak{M}(a, b)$ will be the subset (possibly empty) of $\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2)$ of vector bundles with associated pair (a, b).

The following result was stated and proved by Bohnhorst and Spindler [BS92] for rank-n vector bundles on $\mathbb{P}^n$ with homological dimension 1, but their proof works on $\mathbb{P}^2$ for vector bundles of any rank without modifications.

3.1. Theorem. *For all $(a, b) \in \mathfrak{I}$, the closed set $\overline{\mathfrak{M}(a, b)}$ is an irreducible algebraic subset of $\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2)$ of dimension:*

$$\begin{aligned} \dim \overline{\mathfrak{M}(a, b)} = {} & \dim \operatorname{Hom}(F_1, F_0) + \dim \operatorname{Hom}(F_0, F_1) \\ & - \dim \operatorname{End}(F_1) - \dim \operatorname{End}(F_0) + 1 - \#\{(i, j) : a_i = b_j\}, \end{aligned} \tag{16}$$

where $F_0 = \oplus_{j=1}^{k+r} \mathcal{O}(-b_j)$, $F_1 = \oplus_{i=1}^{k} \mathcal{O}(-a_i)$.

4. Natural pairs and general vector bundles

We say that $(a, b) = ((a_1, \dots, a_k), (b_1, \dots, b_{r+k}))$ is a *natural pair* if it is admissible and

$$b_{r+k} < a_1, \qquad a_k \le b_1 + 2. \tag{17}$$

Through this section, we are going to show that resolutions of general vector bundles have natural pairs:

4.1. Theorem. *One has* $\operatorname{codim} \overline{\mathfrak{M}(a, b)} = 0$ *if and only if (a, b) is a natural pair.*

As a remarkable consequence we will derive a quite simple proof of the irreducibility of the moduli spaces of stable vector bundles on $\mathbb{P}^2$ (other proofs with different techniques can be found in [Bar77a], [Ell83]), [HL93], [LeP79], [Mar78]) and we will compute the regularity and the cohomology of their general elements.

We recall that, since $\dim \operatorname{Ext}^2(\mathcal{F}, \mathcal{F}) = 0$ for any stable vector bundle $\mathcal{F}$ on $\mathbb{P}^2$, the relative moduli $\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2)$ space is smooth of dimension

$$\dim \operatorname{Ext}^1(\mathcal{F}, \mathcal{F}) = 2rc_2 - (r - 1)c_1^2 - r^2 + 1. \tag{18}$$

Let us consider the function $A(t) := h^2(\mathcal{O}(t))$ and the finite differences of first and second order $(\Delta_u A)(t) := A(t+u) - A(t)$ and $(\Delta_v \Delta_u A)(t) := (\Delta_u A)(t+v) - (\Delta_u A)(t)$.

4.2. Lemma. *Let (a, b) be the admissible pair associated to a stable vector bundle $\mathcal{E}$ on $\mathbb{P}^2$. Then*

$$\begin{aligned}\operatorname{codim} \overline{\mathfrak{M}(a,b)} = &\sum_{i=1}^{r} h^1(\mathcal{E}(b_i)) + \#\{(i,j) : a_i = b_j\} \\ &+ \sum_{i,j=1}^{r} (\Delta_{b_{i+r}-a_i} \Delta_{b_{j+r}-a_j} A)(a_i - b_{j+r})\end{aligned} \tag{19}$$

Proof. Let

$$0 \to F_1 \to F_0 \to \mathcal{E} \to 0 \tag{20}$$

be the minimal resolution of $\mathcal{E}$ where

$$F_0 = \bigoplus_{j=1}^{k+r} \mathcal{O}(-b_j), \qquad F_1 = \bigoplus_{i=1}^{k} \mathcal{O}(-a_i). \tag{21}$$

The stability of $\mathcal{E}$ ensures the vanishing $\dim(\operatorname{Ext}^2(\mathcal{E}, \mathcal{E})) = h^2(\mathcal{E}^* \otimes \mathcal{E}) = 0$ so that $h^2({F_0}^* \otimes \mathcal{E}) = h^2({F_1}^* \otimes \mathcal{E})$. Then, from (20) we easily find the following data:

$$\begin{aligned} h^0({F_0}^* \otimes \mathcal{E}) &= h^0({F_0}^* \otimes F_0) - h^0({F_0}^* \otimes F_1), \\ h^0({F_1}^* \otimes \mathcal{E}) &= h^0({F_1}^* \otimes F_0) - h^0({F_1}^* \otimes F_1), \\ \dim(\operatorname{Ext}^1(\mathcal{E}, \mathcal{E})) &= h^1(\mathcal{E}^* \otimes \mathcal{E}) = \\ &= h^1({F_0}^* \otimes \mathcal{E}) - h^1({F_1}^* \otimes \mathcal{E}) + \\ &\quad + h^0({F_1}^* \otimes \mathcal{E}) - h^0({F_0}^* \otimes \mathcal{E}) + 1 \end{aligned} \tag{22}$$

and from (16) we have

$$\begin{aligned} \operatorname{codim} \overline{\mathfrak{M}(a,b)} &= \dim(\operatorname{Ext}^1(\mathcal{E}, \mathcal{E})) - \dim \overline{\mathfrak{M}(a,b)} = \\ &= h^1({F_0}^* \otimes \mathcal{E}) - h^1({F_1}^* \otimes \mathcal{E}) + \#\{(i,j) : a_i = b_j\} \end{aligned} \tag{23}$$

Now, by splitting F_0 as $\mathcal{O}(-b_1) \oplus \cdots \oplus \mathcal{O}(-b_{r-1}) \oplus \tilde{F_0}$ with $\tilde{F_0} := \oplus_{i=r+1}^{k+r} \mathcal{O}(-b_i)$, the above formula becomes

$$\begin{aligned} \operatorname{codim} \overline{\mathfrak{M}(a,b)} &= \sum_{i=1}^{r} h^1(\mathcal{E}(b_i)) + \#\{(i,j) : a_i = b_j\} \\ &\quad + h^1(\tilde{F_0}^* \otimes \mathcal{E}) - h^1({F_1}^* \otimes \mathcal{E}) \\ &= \sum_{i=1}^{r} h^1(\mathcal{E}(b_i)) + \#\{(i,j) : a_i = b_j\} \\ &\quad + h^2(\tilde{F_0}^* \otimes F_1) - h^2(\tilde{F_0}^* \otimes F_0) \\ &\quad - h^2({F_1}^* \otimes F_1) + h^2({F_1}^* \otimes F_0). \end{aligned} \tag{24}$$

Since $h^2(\tilde{F_0}^* \otimes F_0) = h^2(\tilde{F_0}^* \otimes \tilde{F_0})$ and $h^2({F_1}^* \otimes F_0) = h^2({F_1}^* \otimes \tilde{F_0})$ so

$$\begin{aligned} \operatorname{codim} \overline{\mathfrak{M}(a,b)} &= \sum_{i=1}^{r} h^1(\mathcal{E}(b_i)) + \#\{(i,j) : a_i = b_j\} \\ &\quad + h^2(\tilde{F_0}^* \otimes F_1) - h^2(\tilde{F_0}^* \otimes \tilde{F_0}) \\ &\quad - h^2({F_1}^* \otimes F_1) + h^2({F_1}^* \otimes \tilde{F_0}). \end{aligned} \tag{25}$$

Finally, distributing the direct sums appearing in the definition of $\tilde{F}_0$ and F_1 the equation (25) becomes

$$\begin{aligned}\operatorname{codim}\overline{\mathfrak{M}(a,b)} &= \sum_{i=1}^{r} h^1(\mathcal{E}(b_i)) + \#\{(i,j) : a_i = b_j\}\\ &\quad + \sum_{i,j=1}^{r} \Big[h^2(\mathcal{O}(b_{i+r}-a_j)) - h^2(\mathcal{O}(b_{i+r}-b_{j+r}))\\ &\qquad - h^2(\mathcal{O}(a_i-a_j)) + h^2(\mathcal{O}(a_i-b_{j+r}))\Big]\\ &= \sum_{i=1}^{r} h^1(\mathcal{E}(b_i)) + \#\{(i,j) : a_i = b_j\}\\ &\quad + \sum_{i,j=1}^{r} (\Delta_{b_{i+r}-a_i}\Delta_{b_{j+r}-a_j}A)(a_i-b_{j+r})\end{aligned} \tag{26}$$

$\square$

We observe that natural pairs are parametrized by three integers s, k, α such that

$$k \geq 1 \qquad \text{and} \qquad -k+1 \leq \alpha \leq k+r \tag{27}$$

as follows: the pair $(a,b)_{s,k,\alpha}$ corresponding to the triple (s,k,α) is the pair associated to a resolution of the form

$$0 \to \mathcal{O}(-s-1)^k \to \mathcal{O}(-s)^\alpha \oplus \mathcal{O}(-s+1)^{r+k-\alpha} \to \mathcal{E} \to 0 \tag{28}$$

if $\alpha \geq 0$, or of the form

$$0 \to \mathcal{O}(-s-1)^{k+\alpha} \oplus \mathcal{O}(-s)^{-\alpha} \to \mathcal{O}(-s+1)^{r+k} \to \mathcal{E} \to 0 \tag{29}$$

if $\alpha < 0$. We exclude the case $\alpha = -k$ so that s is the regularity of the pair, i.e. $s = \max(a_k - 1, b_{r+k})$.

Proof of theorem 4.1. It can be verified by direct computation from theorem 3.1 that, if $\mathcal{E}$ has natural pair, then the codimension of $\overline{\mathfrak{M}(a,b)}$ is zero. Conversely, let u, v be two non-negative integers. Since all finite difference $(\Delta_u A)(t) := A(t+u) - A(t)$ are non decreasing functions of t, then

$$(\Delta_v \Delta_u A)(t) \geq 0 \tag{30}$$

and by the previous lemma

$$\operatorname{codim}\overline{\mathfrak{M}(a,b)} \geq \sum_{i=1}^{r} h^1(\mathcal{E}(b_i)) + \#\{(i,j) : a_i = b_j\}. \tag{31}$$

If $\operatorname{codim}\overline{\mathfrak{M}(a,b)} = 0$, we have $a_k \leq b_1 + 2$ and $\#\{(i,j) : a_i = b_j\} = 0$, since $h^1(\mathcal{E}(b_1)) = 0$ implies $h^2(F_1(b_1)) = 0$. This forces (a,b) to be a natural pair. $\square$

4.3. Proposition. *Let $\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2)$ be nonempty and*

$$s := \max\{\rho \in \mathbb{Z} : r\rho^2 + 2c_1\rho - r\rho \leq 2c_2 - c_1^2 + c_1 - 1\}, \tag{32}$$

or, equivalently,

$$s := \min\{\rho \in \mathbb{Z} : r\rho^2 + 2c_1\rho + r\rho \geq 2c_2 - c_1^2 - c_1\}. \tag{32bis}$$

If α and k are defined by

$$\begin{aligned}\alpha &:= 2c_2 - c_1^2 + r - rs^2 - 2c_1 s,\\ k &:= (rs + c_1 - r + |\alpha|)/2,\end{aligned} \tag{33}$$

then $(a,b)_{s,k,\alpha}$ is the only natural pair of $\mathfrak{M}_{\mathbb{P}^2}(r, c_1, c_2)$.

Proof. This is a verification; we outline the main steps of the computation. In the first place, one must ensure that the natural pair $(a,b)_{s,k,\alpha}$ is actually associated to vector bundles in $\mathfrak{M}_{\mathbb{P}^2}(r,c_1,c_2)$. This amount to show that from (14) the pair $(a,b)_{s,k,\alpha}$ has the appropriate Chern classes and that conditions (27) hold.

From theorem 4.1, any pair (a,b) such that $\dim \overline{\mathfrak{M}(a,b)} = 0$ is a natural pair of the form $(a,b)_{s,k,\alpha}$. From resolutions (28) and (29) we find that α, k must satisfy (33). Then, it remains to verify that s is uniquely determined from r, c_1, c_2 and satisfy (32). By substitution, the inequalities $-k < \alpha \leq k+r$ turn into

$$rs^2 + 2c_1 s - c_1 - rs + 1 \leq 2c_2 - c_1^2 \leq rs^2 + 2c_1 s + c_1 + rs. \tag{34}$$

Since the intervals $[rs^2 + 2c_1 s - c_1 - rs + 1, \quad rs^2 + 2c_1 s + c_1 + rs]$ are disjoint for s varying in $\mathbb{Z}$, then equations (32) and (32bis) give the only suitable value for s. $\square$

4.4. Theorem. *The moduli spaces of stable rank r vector bundles on $\mathbb{P}^2$ are irreducible.*

Proof. Moduli space of stable rank r vector bundles on $\mathbb{P}^2$ are smooth. By the previous proposition they can have only one connected component. $\square$

4.5. Corollary. *The general element of $\mathfrak{M}_{\mathbb{P}^2}(r,c_1,c_2)$ has natural cohomology.*

The above corollary justify the terminology *"natural pair"*. A different proof for it can be found in [HL93].

Now, we are going to give some inequalities on the regularity and the cohomology of stable vector bundles using proposition 4.3. In particular, for rank 2 vector bundles, the next two corollaries give respectively a refined version of corollary 5.4 [Bru80] and proposition 7.1 [Har78].

4.6. Corollary. *A general vector bundle $\mathcal{E}$ in $\mathfrak{M}_{\mathbb{P}^2}(r,c_1,c_2)$ has regularity* $\operatorname{reg}(\mathcal{E}) = s$*, where s is given by (32).*

4.7. Corollary. *Let $[\mathcal{E}]$ be a vector bundle in $\mathfrak{M} = \mathfrak{M}_{\mathbb{P}^2}(r,c_1,c_2)$ and s defined by (32). Then $\mathrm{H}^0(\mathcal{E}(t)) \neq 0$ if*

$$\begin{array}{ll} t \geq s & \textit{when} \quad rs^2 + 2c_1 s + rs = 2c_2 - c_1^2 - c_1, \\ t \geq s-1 & \textit{otherwise.} \end{array}$$

The above inequality is sharp, in the sense that, if $\mathcal{E}$ is general, it gives a necessary and sufficient condition.

Proof. Let $((a_1,\ldots,a_k),(b_1,\ldots,b_{k+r}))$ be the admissible pair associated to a vector bundle $\mathcal{E}$ in $\mathfrak{M}$. Then, one has $\mathrm{H}^0(\mathcal{E}(t)) \neq 0$ if and only if $t - b_1 \geq 0$. By semicontinuity of cohomology groups and theorem 4.4, it is enough to restrict ourselves to the case where $\mathcal{E}$ is general. So, by (28) and (29) one has $\mathrm{H}^0(\mathcal{E}(t)) \neq 0$ if and only if

$$\begin{array}{ll} t \geq s & \text{if} \quad \alpha = k + r \\ t \geq s-1 & \text{otherwise} \end{array}$$

and the condition $\alpha = k + r$ is equivalent to $rs^2 + 2c_1 s + rs = 2c_2 - c_1^2 - c_1$ by (33). $\square$

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